CHAPTER 1

The Real Numbers

IN THIS CHAPTER we begin the study of the real number system. The concepts discussed here will be used throughout the book.

SECTION 1.1 deals with the axioms that define the real numbers, definitions based on them, and some basic properties that follow from them.

SECTION 1.2 emphasizes the principle of mathematical induction.

SECTION 1.3 introduces basic ideas of set theory in the context of sets of real numbers. In this section we prove two fundamental theorems: the Heine–Borel and Bolzano–Weierstrass theorems.

1.1 THE REAL NUMBER SYSTEM

Having taken calculus, you know a lot about the real number system; however, you probably do not know that all its properties follow from a few basic ones. Although we will not carry out the development of the real number system from these basic properties, it is useful to state them as a starting point for the study of real analysis and also to focus on one property, completeness, that is probably new to you.

Field Properties

The real number system (which we will often call simply the *reals*) is first of all a set $\{a, b, c, \dots\}$ on which the operations of addition and multiplication are defined so that every pair of real numbers has a unique sum and product, both real numbers, with the following properties.

(A) $a + b = b + a$ and $ab = ba$ (commutative laws).

(B) $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ (associative laws).

(C) $a(b + c) = ab + ac$ (distributive law).

(D) There are distinct real numbers 0 and 1 such that $a + 0 = a$ and $a1 = a$ for all a.

(E) For each a there is a real number $-a$ such that $a + (-a) = 0$, and if $a \neq 0$, there is a real number $1/a$ such that $a(1/a) = 1$.

The manipulative properties of the real numbers, such as the relations

$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(3a+2b)(4c+2d) = 12ac + 6ad + 8bc + 4bd,$$
$$(-a) = (-1)a, \quad a(-b) = (-a)b = -ab,$$

and

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad (b, d \neq 0),$$

all follow from **(A)**–**(E)**. We assume that you are familiar with these properties.

A set on which two operations are defined so as to have properties **(A)**–**(E)** is called a *field*. The real number system is by no means the only field. The *rational numbers* (which are the real numbers that can be written as $r = p/q$, where p and q are integers and $q \neq 0$) also form a field under addition and multiplication. The simplest possible field consists of two elements, which we denote by 0 and 1, with addition defined by

$$0 + 0 = 1 + 1 = 0, \quad 1 + 0 = 0 + 1 = 1, \tag{1}$$

and multiplication defined by

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1 \tag{2}$$

(Exercise 2).

The Order Relation

The real number system is ordered by the relation $<$, which has the following properties.

(F) For each pair of real numbers a and b, exactly one of the following is true:

$$a = b, \quad a < b, \quad \text{or} \quad b < a.$$

(G) If $a < b$ and $b < c$, then $a < c$. (The relation $<$ is *transitive*.)

(H) If $a < b$, then $a + c < b + c$ for any c, and if $0 < c$, then $ac < bc$.

A field with an order relation satisfying **(F)**–**(H)** is an *ordered* field. Thus, the real numbers form an ordered field. The rational numbers also form an ordered field, but it is impossible to define an order on the field with two elements defined by (1) and (2) so as to make it into an ordered field (Exercise 2).

We assume that you are familiar with other standard notation connected with the order relation: thus, $a > b$ means that $b < a$; $a \geq b$ means that either $a = b$ or $a > b$; $a \leq b$ means that either $a = b$ or $a < b$; the *absolute value of* a, denoted by $|a|$, equals a if $a \geq 0$ or $-a$ if $a \leq 0$. (Sometimes we call $|a|$ the *magnitude* of a.)

You probably know the following theorem from calculus, but we include the proof for your convenience.

Supremum of a Set

A set S of real numbers is *bounded above* if there is a real number b such that $x \leq b$ whenever $x \in S$. In this case, b is an *upper bound* of S. If b is an upper bound of S, then so is any larger number, because of property **(G)**. If β is an upper bound of S, but no number less than β is, then β is a *supremum* of S, and we write

$$\beta = \sup S.$$

With the real numbers associated in the usual way with the points on a line, these definitions can be interpreted geometrically as follows: b is an upper bound of S if no point of S is to the right of b; $\beta = \sup S$ if no point of S is to the right of β, but there is at least one point of S to the right of any number less than β (Figure 1.1.1).

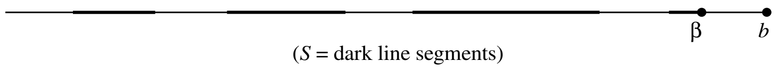

Figure 1.1.1

Example 1.1.1 If S is the set of negative numbers, then any nonnegative number is an upper bound of S, and $\sup S = 0$. If S_1 is the set of negative integers, then any number a such that $a \geq -1$ is an upper bound of S_1, and $\sup S_1 = -1$. ■

This example shows that a supremum of a set may or may not be in the set, since S_1 contains its supremum, but S does not.

A *nonempty* set is a set that has at least one member. The *empty set*, denoted by $\emptyset$, is the set that has no members. Although it may seem foolish to speak of such a set, we will see that it is a useful idea.

The Completeness Axiom

It is one thing to define an object and another to show that there really is an object that satisfies the definition. (For example, does it make sense to define the smallest positive real number?) This observation is particularly appropriate in connection with the definition of the supremum of a set. For example, the empty set is bounded above by every real number, so it has no supremum. (Think about this.) More importantly, we will see in Example 1.1.2 that properties **(A)**–**(H)** do not guarantee that every nonempty set that is bounded above has a supremum. Since this property is indispensable to the rigorous development of calculus, we take it as an axiom for the real numbers.

(I) If a nonempty set of real numbers is bounded above, then it has a supremum.

Property **(I)** is called *completeness*, and we say that the real number system is a *complete ordered field.* It can be shown that the real number system is essentially the only complete ordered field; that is, if an alien from another planet were to construct a mathematical system with properties **(A)**–**(I)**, the alien's system would differ from the real number system only in that the alien might use different symbols for the real numbers and $+$, $\cdot$, and $<$.

Theorem 1.1.3 *If a nonempty set S of real numbers is bounded above, then* $\sup S$ *is the unique real number β such that*

(a) $x \leq \beta$ *for all x in S;*

(b) *if* $\epsilon > 0$ (*no matter how small*), *there is an x_0 in S such that* $x_0 > \beta - \epsilon$.

Proof We first show that $\beta = \sup S$ has properties **(a)** and **(b)**. Since β is an upper bound of S, it must satisfy **(a)**. Since any real number a less than β can be written as $\beta - \epsilon$ with $\epsilon = \beta - a > 0$, **(b)** is just another way of saying that no number less than β is an upper bound of S. Hence, $\beta = \sup S$ satisfies **(a)** and **(b)**.

Now we show that there cannot be more than one real number with properties **(a)** and **(b)**. Suppose that $\beta_1 < \beta_2$ and β_2 has property **(b)**; thus, if $\epsilon > 0$, there is an x_0 in S such that $x_0 > \beta_2 - \epsilon$. Then, by taking $\epsilon = \beta_2 - \beta_1$, we see that there is an x_0 in S such that

$$x_0 > \beta_2 - (\beta_2 - \beta_1) = \beta_1,$$

so β_1 cannot have property **(a)**. Therefore, there cannot be more than one real number that satisfies both **(a)** and **(b)**. □

Some Notation

We will often define a set S by writing $S = \{x \mid \cdots\}$, which means that S consists of all x that satisfy the conditions to the right of the vertical bar; thus, in Example 1.1.1,

$$S = \{x \mid x < 0\} \tag{8}$$

and

$$S_1 = \{x \mid x \text{ is a negative integer}\}.$$

We will sometimes abbreviate "x is a member of S" by $x \in S$, and "x is not a member of S" by $x \notin S$. For example, if S is defined by (8), then

$$-1 \in S \quad \text{but} \quad 0 \notin S.$$

A *nonempty* set is a set that has at least one member. The *empty set* , denoted by $\emptyset$, is the set that has no members. Although it may seem foolish to speak of such a set, we will see that it is a useful concept.

The Archimedean Property

The property of the real numbers described in the next theorem is called the *Archimedean property*. Intuitively, it states that it is possible to exceed any positive number, no matter how large, by adding an arbitrary positive number, no matter how small, to itself sufficiently many times.

Theorem 1.1.4 (The Archimedean Property) *If ρ and ϵ are positive, then $n\epsilon > \rho$ for some integer n.*

Proof The proof is by contradiction. If the statement is false, ρ is an upper bound of the set

$$S = \{x \mid x = n\epsilon, n \text{ is an integer}\}.$$

Therefore, S has a supremum β, by property **(I)**. Therefore,

$$n\epsilon \le \beta \quad \text{for all integers } n. \tag{9}$$

Since $n + 1$ is an integer whenever n is, (9) implies that

$$(n + 1)\epsilon \le \beta$$

and therefore

$$n\epsilon \le \beta - \epsilon$$

for all integers n. Hence, $\beta - \epsilon$ is an upper bound of S. Since $\beta - \epsilon < \beta$, this contradicts the definition of β. □

Density of the Rationals and Irrationals

Definition 1.1.5 A set D is *dense in the reals* if every open interval (a, b) contains a member of D. ■

Theorem 1.1.6 *The rational numbers are dense in the reals; that is, if a and b are real numbers with $a < b$, there is a rational number p/q such that $a < p/q < b$.*

Proof From Theorem 1.1.4 with $\rho = 1$ and $\epsilon = b - a$, there is a positive integer q such that $q(b - a) > 1$. There is also an integer j such that $j > qa$. This is obvious if $a \le 0$, and it follows from Theorem 1.1.4 with $\epsilon = 1$ and $\rho = qa$ if $a > 0$. Let p be the smallest integer such that $p > qa$. Then $p - 1 \le qa$, so

$$qa < p \le qa + 1.$$

Since $1 < q(b - a)$, this implies that

$$qa < p < qa + q(b - a) = qb,$$

so $qa < p < qb$. Therefore, $a < p/q < b$. □

Example 1.1.2 The rational number system is not complete; that is, a set of rational numbers may be bounded above (by rationals), but not have a rational upper bound less than any other rational upper bound. To see this, let

$$S = \left\{ r \,\middle|\, r \text{ is rational and } r^2 < 2 \right\}.$$

If $r \in S$, then $r < \sqrt{2}$. Theorem 1.1.6 implies that if $\epsilon > 0$ there is a rational number r_0 such that $\sqrt{2} - \epsilon < r_0 < \sqrt{2}$, so Theorem 1.1.3 implies that $\sqrt{2} = \sup S$. However, $\sqrt{2}$ is *irrational*; that is, it cannot be written as the ratio of integers (Exercise 3). Therefore, if r_1 is any rational upper bound of S, then $\sqrt{2} < r_1$. By Theorem 1.1.6, there is a rational number r_2 such that $\sqrt{2} < r_2 < r_1$. Since r_2 is also a rational upper bound of S, this shows that S has no rational supremum. ■

Since the rational numbers have properties **(A)**–**(H)**, but not **(I)**, this example shows that **(I)** does not follow from **(A)**–**(H)**.

Theorem 1.1.7 *The set of irrational numbers is dense in the reals; that is, if a and b are real numbers with $a < b$, there is an irrational number t such that $a < t < b$.*

Proof From Theorem 1.1.6, there are rational numbers r_1 and r_2 such that

$$a < r_1 < r_2 < b. \tag{10}$$

Let

$$t = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1).$$

Then t is irrational (why?) and $r_1 < t < r_2$, so $a < t < b$, from (10). □

Infimum of a Set

A set S of real numbers is *bounded below* if there is a real number a such that $x \geq a$ whenever $x \in S$. In this case, a is a *lower bound* of S. If a is a lower bound of S, so is any smaller number, because of property **(G)**. If α is a lower bound of S, but no number greater than α is, then α is an *infimum* of S, and we write

$$\alpha = \inf S.$$

Geometrically, this means that there are no points of S to the left of α, but there is at least one point of S to the left of any number greater than α.

Theorem 1.1.8 *If a nonempty set S of real numbers is bounded below, then* $\inf S$ *is the unique real number α such that*

(a) $x \geq \alpha$ *for all x in S;*

(b) *if $\epsilon > 0$ (no matter how small), there is an x_0 in S such that $x_0 < \alpha + \epsilon$.*

Proof (Exercise 6)

A set S is *bounded* if there are numbers a and b such that $a \leq x \leq b$ for all x in S. A bounded nonempty set has a unique supremum and a unique infimum, and

$$\inf S \leq \sup S \tag{11}$$

(Exercise 7). □