2.3 DIFFERENTIABLE FUNCTIONS OF ONE VARIABLE

In calculus you studied differentiation, emphasizing rules for calculating derivatives. Here we consider the theoretical properties of differentiable functions. In doing this, we assume that you know how to differentiate elementary functions such as x^n, e^x, and $\sin x$, and we will use such functions in examples.

Definition of the Derivative

Definition 2.3.1 A function f is *differentiable* at an interior point x_0 of its domain if the difference quotient

$$\frac{f(x)-f(x_0)}{x-x_0}, \quad x \ne x_0,$$

approaches a limit as x approaches x_0, in which case the limit is called the *derivative of f at x_0*, and is denoted by $f'(x_0)$; thus,

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}. \tag{1}$$

It is sometimes convenient to let $x=x_0+h$ and write (1) as

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}.$$ ■

If f is defined on an open set S, we say that f is *differentiable on S* if f is differentiable at every point of S. If f is differentiable on S, then f' is a function on S. We say that f is *continuously differentiable* on S if f' is continuous on S. If f is differentiable on a neighborhood of x_0, it is reasonable to ask if f' is differentiable at x_0. If so, we denote the derivative of f' at x_0 by $f''(x_0)$. This is the *second derivative of f at x_0*, and it is also denoted by $f^{(2)}(x_0)$. Continuing inductively, if $f^{(n-1)}$ is defined on a neighborhood of x_0, then the nth *derivative of f at x_0*, denoted by $f^{(n)}(x_0)$, is the derivative of $f^{(n-1)}$ at x_0. For convenience we define the *zeroth derivative* of f to be f itself; thus

$$f^{(0)}=f.$$

We assume that you are familiar with the other standard notations for derivatives; for example,

$$f^{(2)}=f'', \quad f^{(3)}=f''',$$

and so on, and

$$\frac{d^n f}{dx^n} = f^{(n)}.$$

Example 2.3.1 If n is a positive integer and

$$f(x) = x^n,$$

then

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^n - x_0^n}{x - x_0} = \frac{x - x_0}{x - x_0} \sum_{k=0}^{n-1} x^{n-k-1} x_0^k,$$

so

$$f'(x_0) = \lim_{x \to x_0} \sum_{k=0}^{n-1} x^{n-k-1} x_0^k = n x_0^{n-1}.$$

Since this holds for every x_0, we drop the subscript and write

$$f'(x) = nx^{n-1} \quad \text{or} \quad \frac{d}{dx}(x^n) = nx^{n-1}. \qquad \blacksquare$$

To derive differentiation formulas for elementary functions such as $\sin x$, $\cos x$, and e^x directly from Definition 2.3.1 requires estimates based on the properties of these functions. Since this is done in calculus, we will not repeat it here.

Interpretations of the Derivative

If $f(x)$ is the position of a particle at time $x \ne x_0$, the difference quotient

$$\frac{f(x) - f(x_0)}{x - x_0}$$

is the average velocity of the particle between times x_0 and x. As x approaches x_0, the average applies to shorter and shorter intervals. Therefore, it makes sense to regard the limit (1), if it exists, as the particle's *instantaneous velocity at time* x_0. This interpretation may be useful even if x is not time, so we often regard $f'(x_0)$ as the *instantaneous rate of change of* $f(x)$ *at* x_0, regardless of the specific nature of the variable x. The derivative also has a geometric interpretation. The equation of the line through two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ on the curve $y = f(x)$ (Figure 2.3.1) is

$$y = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0).$$

Varying x_1 generates lines through $(x_0, f(x_0))$ that rotate into the line

$$y = f(x_0) + f'(x_0)(x - x_0) \tag{2}$$

as x_1 approaches x_0. This is the *tangent* to the curve $y = f(x)$ at the point $(x_0, f(x_0))$. Figure 2.3.2 depicts the situation for various values of x_1.

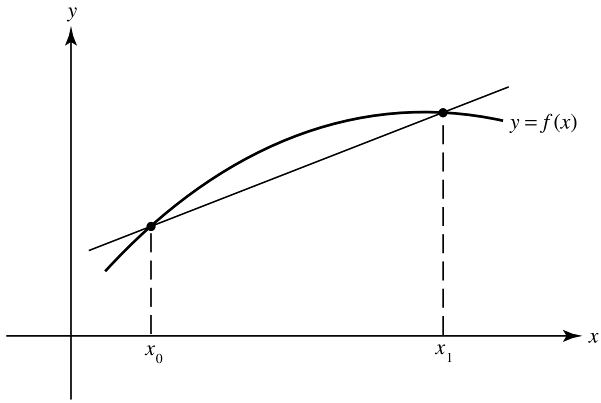

Figure 2.3.1

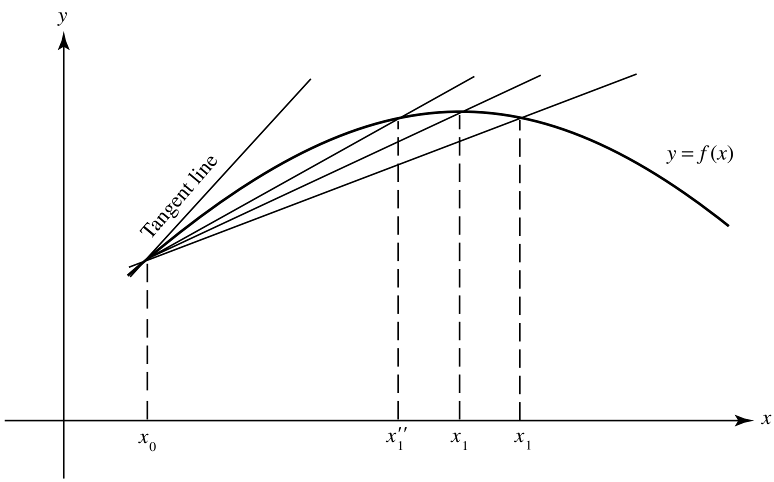

Figure 2.3.2

Here is a less intuitive definition of the tangent line: If the function

$$T(x) = f(x_0) + m(x - x_0)$$

approximates f so well near x_0 that

$$\lim_{x \to x_0} \frac{f(x) - T(x)}{x - x_0} = 0,$$

we say that the line $y = T(x)$ is *tangent to the curve* $y = f(x)$ *at* $(x_0, f(x_0))$.

This tangent line exists if and only if $f'(x_0)$ exists, in which case m is uniquely determined by $m = f'(x_0)$ (Exercise 1). Thus, (2) is the equation of the tangent line.

We will use the following lemma to study differentiable functions.

Lemma 2.3.2 *If f is differentiable at x_0, then*

$$f(x) = f(x_0) + [f'(x_0) + E(x)](x - x_0), \tag{3}$$

where E is defined on a neighborhood of x_0 and

$$\lim_{x \to x_0} E(x) = E(x_0) = 0.$$

Proof Define

$$E(x) = \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0} - f'(x_0), & x \in D_f \text{ and } x \neq x_0, \\ 0, & x = x_0. \end{cases} \tag{4}$$

Solving (4) for $f(x)$ yields (3) if $x \neq x_0$, and (3) is obvious if $x = x_0$. Definition 2.3.1 implies that $\lim_{x \to x_0} E(x) = 0$. We defined $E(x_0) = 0$ to make E continuous at x_0. ☐

Since the right side of (3) is continuous at x_0, so is the left. This yields the following theorem.

Theorem 2.3.3 *If f is differentiable at x_0, then f is continuous at x_0.*

The converse of this theorem is false, since a function may be continuous at a point without being differentiable at the point.

Example 2.3.2 The function

$$f(x) = |x|$$

can be written as

$$f(x) = x, \quad x > 0, \tag{5}$$

or as

$$f(x) = -x, \quad x < 0. \tag{6}$$

From (5),

$$f'(x) = x, \quad x > 0,$$

and from (6),

$$f'(x) = -x, \quad x < 0.$$

Neither (5) nor (6) holds throughout any neighborhood of 0, so neither can be used alone to calculate $f'(0)$. In fact, since the one-sided limits

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{x}{x} \tag{7}$$

are different,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

does not exist (Theorem 2.1.6); thus, f is not differentiable at 0, even though it is continuous at 0. ■

Interchanging Differentiation and Arithmetic Operations

The following theorem should be familiar from calculus.

Theorem 2.3.4 *If f and g are differentiable at x_0, then so are $f+g$, $f-g$, and fg, with*

(a) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$;

(b) $(f-g)'(x_0) = f'(x_0) - g(x_0)$;

(c) $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

The quotient f/g is differentiable at x_0 if $g(x_0) \neq 0$, with

(d) $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}$.

Proof The proof is accomplished by forming the appropriate difference quotients and applying Definition 2.3.1 and Theorem 2.1.4. We will prove **(c)** and leave the rest to you (Exercises 9, 10, and 11).

The trick is to add and subtract the right quantity in the numerator of the difference quotient for $(fg)'(x_0)$; thus,

$$\begin{aligned}\frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} &= \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\ &= \frac{f(x) - f(x_0)}{x - x_0}g(x) + f(x_0)\frac{g(x) - g(x_0)}{x - x_0}.\end{aligned}$$

The difference quotients on the right approach $f'(x_0)$ and $g'(x_0)$ as x approaches x_0, and $\lim_{x \to x_0} g(x) = g(x_0)$ (Theorem 2.3.3). This proves **(c)**. □

The Chain Rule

Here is the rule for differentiating a composite function.

Theorem 2.3.5 (The Chain Rule) *Suppose that g is differentiable at x_0 and f is differentiable at $g(x_0)$. Then the composite function $h = f \circ g$, defined by*

$$h(x) = f(g(x)),$$

is differentiable at x_0, with

$$h'(x_0) = f'(g(x_0))g'(x_0).$$

Proof Since f is differentiable at $g(x_0)$, Lemma 2.3.2 implies that

$$f(t) - f(g(x_0)) = [f'(g(x_0)) + E(t)][t - g(x_0)],$$

where

$$\lim_{t \to g(x_0)} E(t) = E(g(x_0)) = 0. \tag{9}$$

Letting $t = g(x)$ yields

$$f(g(x)) - f(g(x_0)) = [f'(g(x_0)) + E(g(x))][g(x) - g(x_0)].$$

Since $h(x) = f(g(x))$, this implies that

$$\frac{h(x) - h(x_0)}{x - x_0} = [f'(g(x_0) + E(g(x))]\frac{g(x) - g(x_0)}{x - x_0}. \tag{10}$$

Since g is continuous at x_0 (Theorem 2.3.3), (9) and Theorem 2.2.7 imply that

$$\lim_{x \to x_0} E(g(x)) = E(g(x_0)) = 0.$$

Therefore, (10) implies that

$$h'(x_0) = \lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} = f'(g(x_0))g'(x_0),$$

as stated. ☐

Example 2.3.3 If

$$f(x) = \sin x \quad \text{and} \quad g(x) = \frac{1}{x}, \quad x \neq 0,$$

then

$$h(x) = f(g(x)) = \sin\frac{1}{x}, \quad x \neq 0,$$

and

$$h'(x) = f'(g(x))g(x) = \left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right), \quad x \neq 0.$$ ■

It may seem reasonable to justify the chain rule by writing

$$\begin{aligned}\frac{h(x) - h(x_0)}{x - x_0} &= \frac{f(g(x)) - f(g(x_0))}{x - x_0} \\ &= \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)}\,\frac{g(x) - g(x_0)}{x - x_0}\end{aligned}$$

and arguing that

$$\lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} = f'(g(x_0))$$

(because $\lim_{x\to x_0} g(x) = g(x_0)$) and

$$\lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} = g'(x_0).$$

However, this is not a valid proof (Exercise 13).

One-Sided Derivatives

One-sided limits of difference quotients such as (7) and (8) in Example 2.3.2 are called *one-sided* or *right- and left-hand derivatives*. That is, if f is defined on $[x_0, b)$, the *right-hand derivative of f at x_0* is defined to be

$$f'_+(x_0) = \lim_{x\to x_0+} \frac{f(x)-f(x_0)}{x-x_0}$$

if the limit exists, while if f is defined on $(a, x_0]$, the *left-hand derivative of f at x_0* is defined to be

$$f'_-(x_0) = \lim_{x\to x_0-} \frac{f(x)-f(x_0)}{x-x_0}$$

if the limit exists. Theorem 2.1.6 implies that f is differentiable at x_0 if and only if $f'_+(x_0)$ and $f'_-(x_0)$ exist and are equal, in which case

$$f'(x_0) = f'_+(x_0) = f'_-(x_0).$$

In Example 2.3.2, $f'_+(0) = 1$ and $f'_-(0) = -1$.

Example 2.3.4 If

$$f(x) = \begin{cases} x^3, & x \le 0, \\ x^2 \sin \dfrac{1}{x}, & x > 0, \end{cases} \tag{11}$$

then

$$f'(x) = \begin{cases} 3x^2, & x < 0, \\ 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}, & x > 0. \end{cases} \tag{12}$$

Since neither formula in (11) holds for all x in any neighborhood of 0, we cannot simply differentiate either to obtain $f'(0)$; instead, we calculate

$$f'_+(0) = \lim_{x\to 0+} \frac{x^2 \sin 1x - 0}{x-0} = \lim_{x\to 0+} x \sin \frac{1}{x} = 0,$$

$$f'_-(0) = \lim_{x\to 0-} \frac{x^3 - 0}{x-0} = \lim_{x\to 0-} x^2 = 0;$$

hence, $f'(0) = f'_+(0) = f'_-(0) = 0$. ■

This example shows that there is a difference between a one-sided derivative and a one-sided limit of a derivative, since $f'_+(0) = 0$, but, from (12), $f'(0+) = \lim_{x\to 0+} f'(x)$ does not exist. It also shows that a derivative may exist in a neighborhood of a point x_0 ($= 0$ in this case), but be discontinuous at x_0.

Exercise 4 justifies the method used in

Example 2.3.4 to compute $f'(x)$ for $x \neq 0$.

Definition 2.3.6

(a) We say that f is *differentiable on the closed interval* $[a, b]$ if f is differentiable on the open interval (a, b) and $f'_+(a)$ and $f'_-(b)$ both exist.

(b) We say that f is *continuously differentiable on* $[a, b]$ if f is differentiable on $[a, b]$, f' is continuous on (a, b), $f'_+(a) = f'(a+)$, and $f'_-(b) = f'(b-)$. ■

Extreme Values

We say that $f(x_0)$ is a *local extreme value* of f if there is a $\delta > 0$ such that $f(x) - f(x_0)$ does not change sign on

$$(x_0 - \delta, x_0 + \delta) \cap D_f. \tag{13}$$

More specifically, $f(x_0)$ is a *local maximum value* of f if

$$f(x) \le f(x_0) \tag{14}$$

or a *local minimum value* of f if

$$f(x) \ge f(x_0) \tag{15}$$

for all x in the set (13). The point x_0 is called a *local extreme point* of f, or, more specifically, a *local maximum* or *local minimum point* of f.

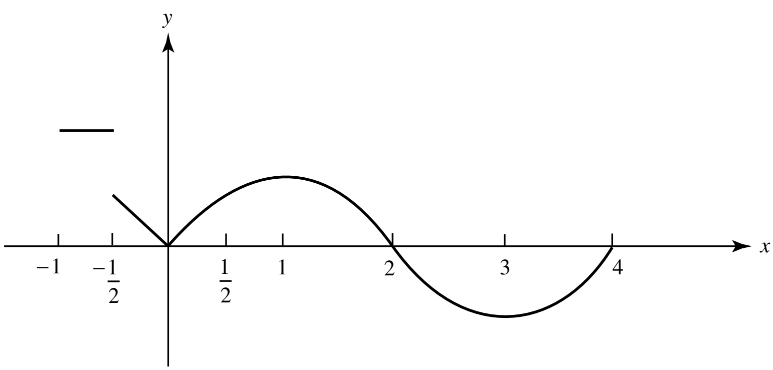

Figure 2.3.3

Example 2.3.5 If

$$f(x) = \begin{cases} 1, & -1 < x \le -\frac{1}{2} \\ |x|, & -\frac{1}{2} < x \le \frac{1}{2}, \\ \dfrac{1}{\sqrt{2}} \sin \dfrac{\pi x}{2}, & \frac{1}{2} < x \le 4 \end{cases}$$

(Figure 2.3.3), then 0, 3, and every x in $(-1, -\frac{1}{2})$ are local minimum points of f, while 1, 4, and every x in $(-1, -\frac{1}{2}]$ are local maximum points. ■

It is geometrically plausible that if the curve $y = f(x)$ has a tangent at a local extreme point of f, then the tangent must be horizontal; that is, have zero slope. (For example, in Figure 2.3.3, see $x = 1$, $x = 3$, and every x in $(-1, -1/2)$.) The following theorem shows that this must be so.

Theorem 2.3.7 *If f is differentiable at a local extreme point $x_0 \in D_f^0$, then $f'(x_0) = 0$.*

If $f'(x_0) = 0$, we say that x_0 is a *critical point* of f. Theorem 2.3.7 says that every local extreme point of f at which f is differentiable is a critical point of f. The converse is false. For example, 0 is a critical point of $f(x) = x^3$, but not a local extreme point.

Rolle's Theorem

The use of Theorem 2.3.7 for finding local extreme points is covered in calculus, so we will not pursue it here. However, we will use Theorem 2.3.7 to prove the following fundamental theorem, which says that if a curve $y = f(x)$ intersects a horizontal line at $x = a$ and $x = b$ and has a tangent at $(x, f(x))$ for every x in (a, b), then there is a point c in (a, b) such that the tangent to the curve at $(c, f(c))$ is horizontal (Figure 2.3.4, page 82).

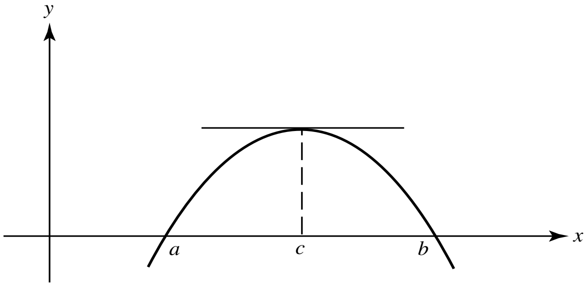

Figure 2.3.4

Theorem 2.3.8 (Rolle's Theorem) *Suppose that f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), and $f(a) = f(b)$. Then $f'(c) = 0$ for some c in the open interval (a,b).*

Proof Since f is continuous on $[a,b]$, f attains a maximum and a minimum value on $[a,b]$ (Theorem 2.2.9). If these two extreme values are the same, then f is constant on (a,b), so $f'(x) = 0$ for all x in (a,b). If the extreme values differ, then at least one must be attained at some point c in the open interval (a,b), and $f'(c) = 0$, by Theorem 2.3.7. □

Intermediate Values of Derivatives

A derivative may exist on an interval $[a,b]$ without being continuous on $[a,b]$. Nevertheless, an intermediate value theorem similar to Theorem 2.2.10 applies to derivatives.

Theorem 2.3.9 (Intermediate Value Theorem for Derivatives) *Suppose that f is differentiable on $[a,b]$, $f'(a) \neq f'(b)$, and μ is between $f'(a)$ and $f'(b)$. Then $f'(c) = \mu$ for some c in (a,b).*

Mean Value Theorems

Theorem 2.3.10 (Generalized Mean Value Theorem) *If f and g are continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then*

$$[g(b)-g(a)]f'(c) = [f(b)-f(a)]g'(c) \tag{20}$$

for some c in (a,b).

Proof The function

$$h(x) = [g(b)-g(a)]f(x) - [f(b)-f(a)]g(x)$$

is continuous on $[a,b]$ and differentiable on (a,b), and

$$h(a) = h(b) = g(b)f(a) - f(b)g(a).$$

Therefore, Rolle's theorem implies that $h'(c) = 0$ for some c in (a,b). Since

$$h'(c) = [g(b)-g(a)]f'(c) - [f(b)-f(a)]g'(c),$$

this implies (20). □

The following special case of Theorem 2.3.10 is important enough to be stated separately.

Theorem 2.3.11 (Mean Value Theorem) *If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then*

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

for some c in (a,b).

Proof Apply Theorem 2.3.10 with $g(x) = x$.

□

Theorem 2.3.11 implies that the tangent to the curve $y = f(x)$ at $(c, f(c))$ is parallel to the line connecting the points $(a, f(a))$ and $(b, f(b))$ on the curve (Figure 2.3.5, page 84).

Consequences of the Mean Value Theorem

If f is differentiable on (a,b) and x_1, $x_2 \in (a,b)$ then f is continuous on the closed interval with endpoints x_1 and x_2 and differentiable on its interior. Hence, the mean value theorem implies that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c between x_1 and x_2. (This is true whether $x_1 < x_2$ or $x_2 < x_1$.) The next three theorems follow from this.

Theorem 2.3.12 *If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b).*

Theorem 2.3.13 *If f' exists and does not change sign on (a, b), then f is monotonic on (a, b) : increasing, nondecreasing, decreasing, or nonincreasing as*

$$f'(x) > 0, \quad f'(x) \geq 0, \quad f'(x) < 0, \quad \textit{or} \quad f'(x) \leq 0,$$

respectively, for all x in (a, b).

Theorem 2.3.14 *If*

$$|f'(x)| \leq M, \quad a < x < b,$$

then

$$|f(x) - f(x')| \leq M|x - x'|, \quad x, x' \in (a, b). \tag{21}$$

A function that satisfies an inequality like (21) for all x and x' in an interval is said to satisfy a *Lipschitz condition* on the interval.

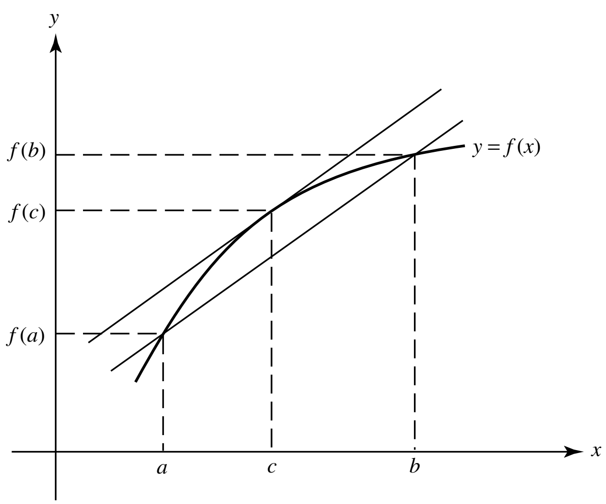

Figure 2.3.5